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THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART III EUCLIDEAN GEOMETRY

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART III

EUCLIDEAN GEOMETRY

A commentary on Chapter III of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus*, paperback edition, (W.A. Banjamin/Addison-Wesley 1973).

R.C. Smith and P. Smith: *Mechanics*, SI edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory by* E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

References to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra 1

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

III.1 INTRODUCTION AND ISOMETRIES OF E^3

Introduction

The aim of this chapter is to prove that a unit-speed curve in E^3 is determined, apart from its position in space, by its curvature and torsion functions, which were introduced in Section II.3; in other words, that the curvature and torsion of a curve, as functions of the arc-length, tell us all there really is to know about it. We cannot begin to prove this until we have decided what we mean by "apart from position in space": when are two curves in different positions in E^3 going to be declared "essentially the same"? We shall not finally answer this question until Section III.5, but we make a start in this section by defining *isometries*, that is mappings of E^3 which preserve distance. You have met some isometries before – translations in *Unit M201 15, Affine Geometry and Convex Cones*, Section 1.1, and orthogonal linear transformations in *Unit M201 24, Orthogonal and Symmetric Transformations*, Section 1. You should remind yourself of the results of the latter section now, as they are used in this section. Apart from that, this section depends only on Sections I.7 and II.1; the later sections of Chapter II are not referred to yet.

READ: Introduction to Chapter III and Section III.1 (pages 98–102).

Comments

(i) *Isometries* An important point about the isometries of E^3 is that they form a group; that is, that they satisfy the following conditions.

(a) The composition of two isometries is again an isometry – this is Lemma 1.3.

(b) Composition of isometries is associative – this is true for composition of any functions.

(c) There is an identity isometry I such that IF = FI = F for all isometries F – this is true because the identity mapping I is an isometry.

(d) Each isometry F has an inverse F^{-1} which is also an isometry. This condition requires slightly more checking. It is an immediate consequence of Definition 1.1 that isometries are one-to-one mappings, for if F(p) = F(q) then

$$d(p, q) = d(F(p), F(q)) = d(F(p), F(p)) = 0$$

and so p = q. The fact that isometries of E^3 are also onto mappings follows from Theorem 1.7, because the translation T is onto and the orthogonal transformation C is onto, so their composite F = TC must also be onto. Since F is one-to-one and onto F has an inverse function $F^{-1} = C^{-1}T^{-1}$, which is a mapping because both C^{-1} and T^{-1} are mappings (see Comments (ii) and (iii)). Now

$$\begin{aligned} d(F^{-1}(p), F^{-1}(q)) &= d(FF^{-1}(p), FF^{-1}(q)) && \text{because F is an isometry} \\ &= d(p, q) && \text{because } FF^{-1} = I \end{aligned}$$

and so F^{-1} is indeed an isometry.

(ii) *Translations* The translations form an important set of isometries, as noted in Example 1.2(1). We shall use the notation T_a to denote translation by a, that is,

$$T_a(p) = p + a.$$

Then

$$T_bT_a(p) = p + a + b = T_{a+b}(p) = T_{b+a}(p)$$

so T_bT_a is translation by $b + a$. In fact the translations correspond to the points of E^3 in such a way that the composition of translations corresponds to ordinary vector addition; hence the commutativity of composition of translations follows from the commutativity of vector addition. We have the following commutative diagram:

$$\begin{tikzcd}
a, b \arrow[r, "+"] \arrow[d, "\text{corresponding translation}"'] & a + b \arrow[d, "\text{corresponding translation}"] \\
T_a, T_b \arrow[r, "\text{composition}"] & T_aT_b = T_{a+b}
\end{tikzcd}$$

Since the identity mapping I is translation through 0, $T_a^{-1} = T_{-a}$ and we have this commutative diagram also:

$$\begin{tikzcd}
a \arrow[r, "\text{additive inverse}"] \arrow[d, "\text{corresponding translation}"'] & -a \arrow[d, "\text{corresponding translation}"] \\
T_a \arrow[r, "\text{inverse}"] & (T_a)^{-1} = T_{-a}
\end{tikzcd}$$

Thus the translations also form a group.

(iii) *Orthogonal Transformations* As noted in Lemma 1.5, these form another important set of isometries. The definition given on page 100 is in fact equivalent to that given in *Unit M201 24, Orthogonal and Symmetric Transformations*, page 8, because all the spaces in which we are interested are finite-dimensional. Thus we can make use of the result proved in that unit, that the following conditions on a linear transformation C of finite-dimensional Euclidean spaces are all equivalent:

(a) C preserves dot products (that is, C is an orthogonal transformation);

(b) The rows of the matrix C are orthonormal;

(c) The columns of the matrix C are orthonormal;

(d) ${}^{t}CC = C{}^{t}C = I$; that is, $C^{-1} = {}^{t}C$ (that is, C is an orthogonal matrix);

(e) C maps some orthonormal basis to another orthonormal basis;

(f) C maps all orthonormal bases to orthonormal bases.

Since, for orthogonal transformations C_1, C_2, C,

$$ {}^{t}(C_1C_2) = {}^{t}C_2{}^{t}C_1 = C_2^{-1}C_1^{-1} = (C_1C_2)^{-1} $$

and

$$ {}^{t}(C^{-1}) = {}^{t}({}^{t}C) = C = (C^{-1})^{-1} $$

this immediately tells us that the composite of two orthogonal transformations is an orthogonal transformation and that the inverse of an orthogonal transformation is an orthogonal transformation. Thus these also form a group.

(iv) *Page 101: Theorem 1.7* This breakdown of an isometry F as TC is very useful. In future work with isometries we shall nearly always use this expression, sometimes without explicitly defining T and C. If $F_1 = T_1C_1$ and $F_2 = T_2C_2$ are isometries then their composite F_1F_2 is also an isometry (Lemma 1.3) so we can write $F_1F_2 = T_3C_3$, where T_3 is the translation part of F_1F_2 and C_3 is the orthogonal part of F_1F_2. Then

$$ T_3C_3 = T_1C_1T_2C_2 $$

but it is *not* generally true that $T_3 = T_1T_2$ (although $C_3 = C_1C_2$) because C_1 and T_2 do not commute in general. (See the Theory Exercises if you are interested in investigating this further.)

Supplementary Comments

(i) *Page 99: rotation* The formulas for the coordinates of q in terms of those of p were worked out in *Unit M201 24, Orthogonal and Symmetric Transformations*, Section 1.2.

(ii) *Page 101: lines −10 and −9* If we put $\mathbf{r} = a\mathbf{p} + b\mathbf{q}$, then $r_i = ap_i + bq_i$ for $i = 1, 2, 3$. Using the identity established in line −11,

$$ F(\mathbf{r}) = \Sigma r_iF(\mathbf{u}_i), \qquad F(\mathbf{p}) = \Sigma p_iF(\mathbf{u}_i), \qquad F(\mathbf{q}) = \Sigma q_iF(\mathbf{u}_i) $$

so

$$ \begin{aligned} F(\mathbf{r}) &= \Sigma r_iF(\mathbf{u}_i) \\ &= \Sigma (ap_i + bq_i)F(\mathbf{u}_i) \\ &= \Sigma ap_iF(\mathbf{u}_i) + \Sigma bq_iF(\mathbf{u}_i) \\ &= a\Sigma p_iF(\mathbf{u}_i) + b\Sigma q_iF(\mathbf{u}_i) \\ &= aF(\mathbf{p}) + bF(\mathbf{q}). \end{aligned} $$

Summary

Notation

F	Page 98, Definition 1.1
T	Page 98, Example 1.2(1)
$T_{\mathbf{a}}$	Text, page 6
C	Page 100, line 11
I	Page 99, line −6

Definitions

(i)	Isometry F	Page 98, Definition 1.1
(ii)	Translation by **a**, $T_{\mathbf{a}}$	Page 98, Example 1.2(1)
(iii)	Orthogonal transformation C	Page 100, line 10
(iv)	Identity mapping I	Page 99, line −6

Example

(i)	Rotation about the z axis	Page 99, Example 1.2(2)

Results

(i)	The composite of two isometries is again an isometry.	Page 99, Lemma 1.3
(ii)	For translations:	
	$T_{\mathbf{a}}T_{\mathbf{b}} = T_{\mathbf{b}}T_{\mathbf{a}} = T_{\mathbf{a}+\mathbf{b}}$;	Page 99, Lemma 1.4(1)
	$(T_{\mathbf{a}})^{-1} = T_{-\mathbf{a}}$.	Page 100, Lemma 1.4(2)
(iii)	Given **p**, **q** in E^3, there is a unique translation T such that T(**p**) = **q**.	Page 101, Lemma 1.4(3)
(iv)	If T is a translation and T(**p**) = **p** for any **p**, then T = I.	Page 100, lines 8–9
(v)	Translations are isometries.	Page 98, Example 1.2(1)
(vi)	Orthogonal transformations are isometries.	Page 100, Lemma 1.5
(vii)	Any isometry fixing **0** is an orthogonal transformation.	Page 100, Lemma 1.6
(viii)	Each isometry F has a unique expression as TC, where T is a translation and C is an orthogonal transformation. Moreover $T = T_{F(\mathbf{0})}$.	Page 101, Theorem 1.7
(ix)	Every isometry has an inverse isometry.	Text, page 5.

Techniques

(i)	Recognition of isometries, translations, orthogonal transformations.	Page 98, Definition 1.1, Page 98, Example 1.2(1), and Page 100, line 10.
(ii)	Expression of an isometry in standard form TC.	Page 101, Theorem 1.7
(iii)	Calculation of the image of a point under an isometry expressed in standard form.	Page 102, line −5

Exercises

Technique (i)

1. Page 103, Exercise 4.
2. Page 103, Exercise 6, (a) – (d), first part.

Technique (ii)

3. Page 103, Exercise 6, second part.

Technique (iii)

4. Page 103, Exercise 5(a).

Theory Exercises (omit if short of time)

5. Page 103, Exercise 1.
6. Page 103, Exercise 2.
7. Page 103, Exercise 3.

Solutions

1. Page 103, Exercise 4.

As noted in Comment (iii), C is orthogonal $\Longleftrightarrow C^tC = I \Longleftrightarrow$ the rows of C are orthonormal. We check the last condition as it involves only six calculations.

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\cdot\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \frac{1}{9}(4+4+1) = 1$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{1}{9}(4+1+4) = 1$$

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{9}(1+4+4) = 1$$

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{1}{9}(-4+2+2) = 0$$

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{9}(-2+4-2) = 0$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{9}(2+2-4) = 0$$

This shows that C is orthogonal.

$$C(p) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{19}{3} \\ -\frac{7}{3} \end{pmatrix}$$

$$C(q) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ -\frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$$

$p \cdot q = 3.1 + 1.0 + (-6)(3) = 3 - 18 = -15.$

$C(p) \cdot C(q) = \left(\frac{2}{3}\right)\left(-\frac{5}{3}\right) + \left(\frac{19}{3}\right)\left(-\frac{4}{3}\right) + \left(-\frac{7}{3}\right)\left(\frac{7}{3}\right) = \frac{1}{9}(-10 - 76 - 49) = -\frac{135}{9} = -15.$

2. Page 103, Exercise 6.

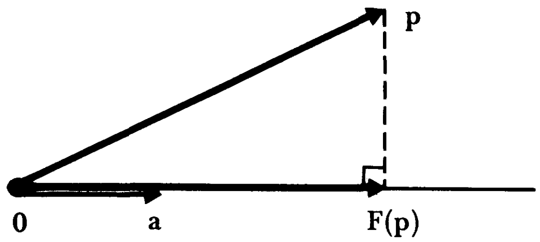
In each case we must check whether $d(F(p), F(q)) = d(p, q)$ for all p and q.

(a) $F(p) = -p$.

$$\begin{aligned} d(F(p), F(q)) &= d(-p, -q) \\ &= \|-p - (-q)\| \\ &= \|(-1)(p - q)\| \\ &= |-1| \, \|p - q\| && \text{(using } \|ap\| = |a| \, \|p\|\text{)} \\ &= 1 \, \|p - q\| \\ &= d(p, q). \end{aligned}$$

This is true for all p and q, so F is an isometry.

(b) In this case $F: p \longmapsto p \cdot a \, a$ is just the perpendicular projection of E^3 onto the one-dimensional subspace spanned by a. We can easily see from the picture



that F does not preserve norms: for example, we can choose p to be orthogonal to a and then $F(p) = 0$ so $\|F(p)\| \neq \|p\|$. Since F does not preserve norms, F cannot preserve dot products.

Alternatively we proceed as follows.

$$\begin{aligned} d(F(p), F(q)) &= d(p \cdot a \, a, q \cdot a \, a) \\ &= \|p \cdot a \, a - q \cdot a \, a\| \\ &= \|(p \cdot a - q \cdot a) a\| \\ &= |p \cdot a - q \cdot a| \, \|a\|, && \text{as above,} \\ &= |p \cdot a - q \cdot a|, && \text{since } \|a\| = 1, \\ &= |(p - q) \cdot a| \\ &= \|p - q\| \, \|a\| \, |\cos \vartheta|, && \text{where } \vartheta \text{ is the angle between } p - q \text{ and } a, \\ &= d(p, q) \, |\cos \vartheta|, && \text{since } \|a\| = 1, \end{aligned}$$

which is not equal to $d(p, q)$ unless ϑ is a multiple of π.

As yet another approach, since the image of F is not the whole of E^3, F is not onto E^3 and so F cannot be an isometry, by Comment (i).

(c) $F(p) = (p_3 - 1, p_2 - 2, p_1 - 3)$.

$$\begin{aligned} d(F(p), F(q)) &= d((p_3 - 1, p_2 - 2, p_1 - 3), (q_3 - 1, q_2 - 2, q_1 - 3)) \\ &= \|(p_3 - 1, p_2 - 2, p_1 - 3) - (q_3 - 1, q_2 - 2, q_1 - 3)\| \\ &= \|(p_3 - q_3, p_2 - q_2, p_1 - q_1)\| \\ &= ((p_3 - q_3)^2 + (p_2 - q_2)^2 + (p_1 - q_1)^2)^{\frac{1}{2}} \\ &= \|(p_1 - q_1, p_2 - q_2, p_3 - q_3)\| \\ &= \|p - q\| \\ &= d(p, q). \end{aligned}$$

This is true for all **p** and **q**, and so F is an isometry.

(d) $F(p) = (p_1, p_2, 1)$.

$$\begin{aligned} d(F(p), F(q)) &= d((p_1, p_2, 1), (q_1, q_2, 1)) \\ &= \|(p_1, p_2, 1) - (q_1, q_2, 1)\| \\ &= \|(p_1 - q_1, p_2 - q_2, 0)\| \\ &= ((p_1 - q_1)^2 + (p_2 - q_2)^2)^{\frac{1}{2}}, \end{aligned}$$

which is not equal to d(p, q) unless $p_3 = q_3$. For example,

$$d((0, 0, 0), (0, 0, 1)) = 1 \quad \text{but}$$

$$d(F(0, 0, 0), F(0, 0, 1)) = d((0, 0, 1), (0, 0, 1)) = 0.$$

Thus F is not an isometry.

Again, we could note that, since the image of F is

$$\{ p \in E^3 : p_3 = 1 \} \neq E^3,$$

F is not onto, and so cannot be an isometry.

3. Page 103, Exercise 6.

(a) $F(p) = -p$ and so $F = -I$, which is an orthogonal transformation. This is already in standard form, and so T is the identity, $C = -I$.

(b) F is not an isometry.

(c) $F(0) = (-1, -2, -3)$ so T, the translation part of F, is translation by $(-1, -2, -3)$. If $F = TC$,

$$F(p) = TC(p)$$

i.e. $(p_3 - 1, p_2 - 2, p_1 - 3) = C(p) + (-1, -2, -3)$

i.e. $(p_3, p_2, p_1) + (-1, -2, -3) = C(p) + (-1, -2, -3)$

so $C(p) = (p_3, p_2, p_1)$

so C has matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(In fact this represents reflection in the plane x = z.)

(d) F is not an isometry.

4. Page 103, Exercise 5(a).

$F(p) = T_a C(p) = a + C(p)$

$$= \begin{pmatrix}1\\3\\-1\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2\\-2\\8\end{pmatrix}$$

$$= \begin{pmatrix}1\\3\\-1\end{pmatrix} + \begin{pmatrix}-\frac{6}{\sqrt{2}}\\-2\\\frac{10}{\sqrt{2}}\end{pmatrix}$$

$$= \begin{pmatrix}1 - 3\sqrt{2}\\1\\-1 + 5\sqrt{2}\end{pmatrix}$$

5. Page 103, Exercise 1.

Put $F = CT_a$. Since C and T_a are isometries F is an isometry (by Lemma III.1.3), and thus by Theorem III.1.7 we can write $F = T_b D$ where T_b is translation through b and D is an orthogonal transformation. Now

$$CT_a(p) = T_b D(p) \qquad \text{for all } p$$

i.e.

$$C(a + p) = b + D(p)$$

i.e.

$$C(a) + C(p) = b + D(p).$$

Putting $p = 0$ gives $C(a) = b$ because $C(0) = D(0) = 0$.

Substituting back, we have

$$b + C(p) = b + D(p) \qquad \text{for all } p$$

i.e.

$$C(p) = D(p) \qquad \text{for all } p$$

and so

$$C = D.$$

Thus

$$F = T_b D = T_{C(a)} C.$$

Alternatively, we could show that CT_a and $T_{C(a)}C$ have the same effect on every point of E^3. For p in E^3,

$$CT_a(p) = C(a + p) = C(a) + C(p),$$

while

$$T_{C(a)}C(p) = T_{C(a)}(C(p)) = C(a) + C(p).$$

6. Page 103, Exercise 2.

$$F = T_aA \quad \text{and} \quad G = T_bB, \quad \text{so}$$

$$FG = T_aAT_bB.$$

By the result of Exercise III.1.1,

$$AT_b = T_{A(b)}A$$

and so

$$FG = T_aAT_bB = T_aT_{A(b)}AB.$$

$T_aT_{A(b)}$ is the translation through $a + A(b)$; AB is an orthogonal transformation, being the product of two orthogonal transformations. Thus FG has translation part $T_{a + A(b)}$ and orthogonal part AB.

Similarly GF has translation part $T_{b + B(a)}$ and orthogonal part BA.

7. Page 103, Exercise 3.

$F = T_aC$. Since T_a and C both have inverses we know that

$$F^{-1} = (T_aC)^{-1} = C^{-1}T_a^{-1}.$$

Now

$$\begin{aligned} C^{-1}T_a^{-1} &= C^{-1}T_{-a}, && \text{by Lemma III.1.4(2),} \\ &= T_{C^{-1}(-a)}C^{-1}, && \text{by Exercise III.1.1,} \\ &= T_{-C^{-1}(a)}C^{-1}, && \text{by linearity of } C^{-1}. \end{aligned}$$

Thus the translation part of F^{-1} is $T_{-C^{-1}(a)}$; the orthogonal part of F^{-1} is C^{-1}.

III.2 THE DERIVATIVE MAP OF AN ISOMETRY

Introduction

This section follows on from Sections III.1 and I.7. We also need to use certain results from M201 – that there exists a unique linear transformation taking one basis of E^3 to another basis of E^3 (see *Unit M201 2, Linear Transformations*, Section 1.8) and that a linear transformation is orthogonal if and only if it takes an orthonormal basis to another orthonormal basis (see *Unit M201 24, Orthogonal and Symmetric Transformations*, Section 1.2).

The purpose of this short section is to show that there is a unique isometry taking any given *frame* in E^3 to any other given frame in E^3. In order to do this we have to look at the derivative map F_*: we find that this takes a particularly simple form when F is an isometry, for if $F = TC$ then F_* is represented simply by C.

READ: Section III.2 (pages 104–106).

Comment

(i) *Page 105: lines -18 to -15* F_{*p} and C cannot be the same function, for they act on different spaces – $T_p(E^3)$ and E^3 respectively. However, $T_p(E^3)$ is isomorphic to E^3 via the "canonical" isomorphism $\phi_p\colon E^3 \longrightarrow T_p(E^3)$ given by

$$\phi_p\colon v \longmapsto v_p \qquad (v \in E^3).$$

To say that F_{*p} and C "differ only by the canonical isomorphisms of E^3", means, in effect, that the following diagram commutes; that is, that F_{*p} and C send vectors that correspond to each other via ϕ_p to vectors that correspond to each other via $\phi_{F(p)}$.

$$\begin{tikzcd} E^3 \arrow[r, "C"] \arrow[d, "\phi_p"'] & E^3 \arrow[d, "\phi_{F(p)}"] \\ T_p(E^3) \arrow[r, "F_{*p}"] & T_{F(p)}(E^3) \end{tikzcd}$$

$$\begin{tikzcd} v \arrow[r, "C"] \arrow[d, "\phi_p"'] & Cv \arrow[d, "\phi_{F(p)}"] \\ v_p \arrow[r, "F_{*p}"] & F_{*p}(v_p) = Cv_{F(p)} \end{tikzcd}$$

Supplementary Comment

(i) *Page 105, line 4* If $v = (v_1, v_2, v_3)$ then the ith coordinate of Cv is $\sum_j c_{ij}v_j$, so that

$$(Cv)_q = \sum_i\sum_j c_{ij}v_jU_i(q)$$

for any point q. In particular, putting $q = F(p)$ gives

$$(Cv)_{F(p)} = \sum_i\sum_j c_{ij}v_jU_i(F(p)).$$

But $F_*(v_p) = (Cv)_{F(p)}$, and so

$$F_*(v_p) = \sum_i\sum_j c_{ij}v_jU_i(F(p)) = \sum_{i,j} c_{ij}v_jU_i(F(p)).$$

However, $v_p = \sum_j v_jU_j(p)$, so this becomes

$$F_*\left(\sum_j v_jU_j(p)\right) = \sum_{i,j} c_{ij}v_jU_i(F(p)),$$

which can be abbreviated to the result given in line 4 by omitting the "p"s and writing $\bar{U}_i$ for $U_i(F)$.

Summary

Results

(i)	The derivative map of an isometry $F = TC$ sends the tangent vector v at the point p to the tangent vector Cv at the point $F(p)$.	Page 104, Theorem 2.1
(ii)	If F is an isometry, F_{*p} is an orthogonal transformation from $T_p(E^3)$ to $T_{F(p)}(E^3)$.	Page 105, Corollary 2.2
(iii)	Given two frames in E^3, there is a unique isometry taking one onto the other.	Page 105, Theorem 2.3

Techniques

(i)	Finding the derivative map of an isometry using Result (i).	Page 104, Theorem 2.1
(ii)	Finding the isometry mentioned in Result (iii), using the method employed in the proof of Theorem 2.3.	Page 105, line −3 of text

Exercises

Technique (i)

1. Page 106, Exercise 1.

Technique (ii)

2. Page 107, Exercise 5.

Theory Exercises (omit if short of time)

3. Page 106, Exercise 2. Write $F = T_aA$ and $G = T_bB$ and use the results of Exercise III.1.2, whose solution occurs on Text page 13, and Exercise III.1.3, whose solution occurs on Text page 13.

4. Page 107, Exercise 4(a). (HINT: $\mathbf{r}$ is in the plane through $\mathbf{p}$ orthogonal to $\mathbf{q}$ if and only if $(\mathbf{r} - \mathbf{p})\cdot\mathbf{q} = 0$.)

Solutions

1. Page 106, Exercise 1.

We can write T = TI for any isometry T, where I is the identity mapping. In the case when T is itself a translation this expresses T in standard form, as I is an orthogonal transformation. Using Theorem III.2.1 we have

$$\begin{aligned} T_*(v) &= (Iv)_{T(p)} \\ &= v_{T(p)}, \qquad \text{because } Iv = v. \end{aligned}$$

Thus $T_*(v)$ has the same Euclidean coordinates as v; only the point of application has changed. In other words, $T_*(v)$ is parallel to v.

2. Page 107, Exercise 5.

The attitude matrix of the **e** frame is

$$A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and the attitude matrix of the **f** frame is

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

By the argument at the end of the text on page 106 we must have that

$$C = {}^tBA = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}$$

and T is the translation through

$$q - C(p) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{3\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix}$$

Thus the isometry F is given by

$$F(r) = \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

for all points $r = (r_1, r_2, r_3)$ of E^3.

Let us check that F really does carry the e frame into the f frame; that is, $F_*(e_{ip}) = f_{iq}$ for i = 1, 2, 3; that is, $F(p) = q$ and $Ce_i = f_i$ for i = 1, 2, 3.

$$F(p) = \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1-\frac{2\sqrt{2}}{3} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{3\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = q.$$

$$Ce_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = f_1$$

$$Ce_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = f_2$$

$$Ce_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = f_3$$

3. Page 106, Exercise 2.

If $F = T_aA$ and $G = T_bB$, where A, B are the orthogonal parts of F and G, then, by the result of Exercise III.1.2, GF has orthogonal part BA. Hence, by Theorem III.2.1, if v_p is any tangent vector,

$$(GF)_*(v_p) = (BAv)_{GF(p)}.$$

The same theorem gives

$$F_*(v_p) = Av_{F(p)}$$

and, applied to the isometry G and the tangent vector $F_*(v_p)$,

$$G_*(F_*(v_p)) = B(Av)_{G(F(p))},$$

that is

$$(G_*F_*)(v_p) = (BAv)_{GF(p)}.$$

Hence $(GF)_*(v_p) = (G_*F_*)(v_p)$ for all tangent vectors v_p, and so it follows that $(GF)_* = G_*F_*$.

By the result of Exercise III.1.3, F^{-1} has orthogonal part A^{-1}, and so Theorem III.2.1 gives

$$(F^{-1})_*(v_p) = (A^{-1}v)_{F^{-1}(p)}$$

for any tangent vector v_p. We must show that this is the same as $(F_*)^{-1}(v_p)$. We know that

$$F_*(v_p) = (Av)_{F(p)},$$

that is, F_* acts on the vector part of v_p by the orthogonal transformation A and on the point of application of v_p by F. Thus $(F_*)^{-1}$, which is the inverse of F_*, must act on the vector part of v_p by A^{-1} and on the point of application of v_p by F^{-1}; that is,

$$(F_*)^{-1}(v_p) = (A^{-1}v)_{F^{-1}(p)}.$$

Consequently

$$(F_*)^{-1}(v_p) = (F^{-1})_*(v_p)$$

for all v_p, and so $(F_*)^{-1} = (F^{-1})_*$.

4. Page 107, Exercise 4(a).

r is in the plane through p orthogonal to q if and only if

$$(r - p)\cdot q = 0.$$

C preserves dot products so this happens if and only if

$$(C(r - p))\cdot C(q) = 0.$$

Suppose T is translation by the vector a.

C is linear, so

$$\begin{aligned} C(r - p) &= C(r) - C(p) \\ &= C(r) + a - (C(p) + a), \\ &= F(r) - F(p). \end{aligned}$$

Thus

$$(C(r - p))\cdot C(q) = (F(r) - F(p))\cdot C(q),$$

which is zero if and only if F(r) is in the plane through F(p) orthogonal to C(q). Hence F carries the plane through p orthogonal to q to the plane through F(p) orthogonal to C(q).

III.3 ORIENTATION

Introduction

This section continues from Section III.2, also tying in ideas from *Unit M201 24, Orthogonal and Symmetric Transformations*, Sections 2.2 to 2.4. You may recall from that unit that frames in E^3 are of two distinct types, known as left-handed and right-handed, distinguished by which hand "best fits" the frames. Moreover, orthogonal transformations of E^3 also fall into two distinct sets: there are the *proper rotations* (that is, those transformations physically realizable in 3-space as a rotation about an axis) and the *improper rotations* (that is, those transformations which are not realizable as an actual movement in 3-space because they consist of a proper rotation followed by a reflection). It turns out that proper rotations preserve the "handedness" of frames, whereas improper rotations interchange left-handed and right-handed frames, and that the corresponding matrices have determinant +1 for a proper rotation and −1 for an improper rotation.

In this section we give a more mathematical definition of "right-handed" and "left-handed" and classify all isometries of E^3 (not just orthogonal transformations) in terms of their effect upon frames. We are able to state and prove the results quite precisely, in a form that makes them suitable for use in proving the important theorems of Section III.5.

READ: Sections III.3 (pages 107–110).

Comment

(i) *Page 107: Remark 3.1*

(1) In this case the attitude matrix is I, and det I = +1.

(2) Since $\mathbf{e}_3$ is a unit vector orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$, it must be either $\mathbf{e}_1 \times \mathbf{e}_2$ or $-(\mathbf{e}_1 \times \mathbf{e}_2)$. By the result of Exercise II.1.4(d), $\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$, which is +1 if $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$, −1 if $\mathbf{e}_3 = -(\mathbf{e}_1 \times \mathbf{e}_2)$.

(3) This is a restatement of part of Lemma II.1.A, on Text, Part II, page 7.

Supplementary Comments

(i) *Page 110: the determinant* Remember that a formal expansion of

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

yields

$$(v_2w_3 - v_3w_2)e_1 + (v_3w_1 - v_1w_3)e_2 + (v_1w_2 - v_2w_1)e_3.$$

(ii) *Page 110: proof of Lemma 3.5* If the frame is negatively oriented, the e_1 component of $v \times w$ is

$$\begin{aligned} v_2e_2 \times w_3e_3 + v_3e_3 \times w_2e_2 &= (v_2w_3 - v_3w_2)e_2 \times e_3 \\ &= (v_2w_3 - v_3w_2)(-e_1), \qquad \text{by Remark 3.1(3),} \\ &= \epsilon\,(v_2w_3 - v_3w_2)e_1 \end{aligned}$$

because $\epsilon = -1$ when the frame is negatively oriented.

(iii) *Page 110: end of proof of Theorem 3.6*

$$F_*(v) \times F_*(w) = \epsilon \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \qquad \text{by Lemma III.3.5,}$$

$$= \epsilon\,((v_2w_3 - v_3w_2)e_1 + \ldots)$$

$$= \epsilon\,((v_2w_3 - v_3w_2)F_*(U_1(p)) + \ldots)$$

$$= \epsilon\,(F_*((v_2w_3 - v_3w_2)U_1(p) + \ldots)), \qquad \text{because } F_* \text{ is linear,}$$

$$= \epsilon \left(F_* \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right)$$

$$= \epsilon\, F_*(v \times w), \qquad \text{by the definition of cross product,}$$

and

$$\begin{aligned} \epsilon = e_1 \cdot e_2 \times e_3 &= F_*(U_1(p)) \cdot F_*(U_2(p)) \times F_*(U_3(p)) \\ &= (\operatorname{sgn} F)U_1(p) \cdot U_2(p) \times U_3(p), \qquad \text{by Lemma III.3.2,} \\ &= \operatorname{sgn} F, \end{aligned}$$

because the frame $U_1(p)$, $U_2(p)$, $U_3(p)$ is positively oriented.

Summary

Notation

sgn F	Page 108, line 8

Definitions

(i)	Positively oriented Right-handed	Page 107, line −17
(ii)	Negatively oriented Left-handed	Page 107, line −16

(iii)	Sign of an isometry, sgn F	Page 108, line 6
(iv)	Orientation-preserving	Page 109, Definition 3.3
(v)	Orientation-reversing	Page 109, Definition 3.3

Results

(i)	The natural frame field is positively oriented.	Page 107, Remark 3.1(1)
(ii)	The frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is positively oriented if and only if $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.	Page 107, Remark 3.1(2)
(iii)	Frenet frames are positively oriented.	Page 107, Remark 3.1(2)
(iv)	If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a positively oriented frame, $\mathbf{e}_i = \mathbf{e}_j \times \mathbf{e}_k = -\mathbf{e}_k \times \mathbf{e}_j$ for $(i, j, k) = (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$.	Page 107, Remark 3.1(3)
(v)	$F_*(\mathbf{e}_1)\cdot F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) = \operatorname{sgn} F\ \mathbf{e}_1\cdot\mathbf{e}_2 \times \mathbf{e}_3$.	Page 108, Lemma 3.2
(vi)	The coordinates of tangent vectors in terms of a positively oriented frame give the expected formula for the cross product; if the frame is negatively oriented the usual formula must be multiplied by -1.	Page 110, Lemma 3.5
(vii)	Orientation-preserving isometries preserve cross products; orientation-reversing isometries change their sign: $F_*(\mathbf{v} \times \mathbf{w}) = \operatorname{sgn} F\ F_*(\mathbf{v}) \times F_*(\mathbf{w})$.	Page 110, Theorem 3.6

Techniques

(i)	Distinguishing positively and negatively oriented frames.	Page 107, lines $-17, -16$
(ii)	Finding the sign of an isometry	Page 108, lines 6 to 8
(iii)	Finding the effect of an isometry on a cross product, using Result (vii).	Page 110, Theorem 3.6

Exercises

Technique (i)

1. Find the orientation of each of the frames in Exercise III.2.5 on page 107.

Technique (ii)

2. Find the sign of the isometry in Exercise III.2.5 and check that Lemma III.3.2 is satisfied.

Technique (iii)

3. Page 111, Exercise 3.

Theory Exercises (omit if short of time)

4. Page 111, Exercise 1. (HINT: Use the result of Exercise III.1.2.)

5. Page 111, Exercise 2.

Solutions

1. The attitude matrix of the **e** frame is

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

whose determinant is

$$\frac{2}{3}\left(\frac{1}{3}.\frac{2}{3} - \frac{2}{3}.\left(-\frac{2}{3}\right)\right) + \frac{2}{3}\left(\frac{2}{3}.\frac{1}{3} - \left(-\frac{2}{3}\right).\frac{2}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}.\left(-\frac{2}{3}\right) - \frac{1}{3}.\frac{1}{3}\right)$$

$$= \frac{1}{27}(2(2 + 4) + 2(2 + 4) + (4 - 1))$$

$$= 1,$$

so the **e** frame is positively oriented. Alternatively (and more simply)

$$\mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{9}(2.2 - 1.1,\ 1.-2 - 2.2,\ 2.1 - 2.(-2))$$

$$= \frac{1}{9}(3, -6, 6) = \mathbf{e}_3$$

and so the **e** frame is positively oriented by Remark 3.1(2). Similarly,

$$\mathbf{f}_1 \times \mathbf{f}_2 = \frac{1}{\sqrt{2}}(0.0 - 1.1,\ 1.0 - 1.0,\ 1.1 - 0.0)$$

$$= \frac{1}{\sqrt{2}}(-1, 0, 1) = -\mathbf{f}_3$$

and so the **f** frame is negatively oriented.

2. Lemma III.3.2 is satisfied if $F_*(\mathbf{e}_1)\cdot F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) = \operatorname{sgn} F\ \mathbf{e}_1\cdot\mathbf{e}_2 \times \mathbf{e}_3$. We found in the previous section that

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}.$$

$$\operatorname{sgn} F = \det C = \frac{1}{\sqrt{2}}\left(\frac{1}{3}\cdot - \frac{1}{3\sqrt{2}} - \frac{2}{3}\cdot\frac{4}{3\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{2}{3}\cdot\frac{4}{3\sqrt{2}} - \frac{1}{3}\cdot\frac{1}{3\sqrt{2}}\right)$$

$$= \frac{1}{18}(-1 - 8 - 8 - 1) = -1.$$

Since sgn F = −1, Lemma III.3.2 predicts that the **e** frame and the F(**e**) frame (that is the **f** frame) have opposite orientations: we have already shown that this is the case in the previous exercise.

3. Page 111, Exercise 3.

We want to show that $C_*(\mathbf{v} \times \mathbf{w}) = \operatorname{sgn} C\, C_*(\mathbf{v}) \times C_*(\mathbf{w})$.

$$\mathbf{v} = (3, 1, -1)$$

$$\mathbf{w} = (-3, -3, 1)$$

so

$$\mathbf{v} \times \mathbf{w} = (1 - 3, 3 - 3, -9 + 3) = (-2, 0, -6).$$

C is its own orthogonal part, so C_* is represented by the same matrix as C.

$$C_*(\mathbf{v}) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$C_*(\mathbf{w}) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{11}{3} \\ -\frac{7}{3} \end{pmatrix}$$

$$C_*(\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ \frac{8}{3} \\ -\frac{14}{3} \end{pmatrix}$$

$$C_*(\mathbf{v}) \times C_*(\mathbf{w}) = (-1, 3, 1) \times \left(-\frac{1}{3}, -\frac{11}{3}, -\frac{7}{3}\right)$$

$$= \left(-\frac{10}{3}, -\frac{8}{3}, \frac{14}{3}\right) = -C_*(\mathbf{v} \times \mathbf{w})$$

so the formula is checked if sgn C = −1.

However,

$$\operatorname{sgn} C = \det C = -\frac{2}{3}\left(\frac{1}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{2}{3}\right) + \frac{2}{3}\left(-\frac{2}{3}\cdot\frac{1}{3} - \frac{2}{3}\cdot\frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\cdot\frac{2}{3} - \frac{1}{3}\cdot\frac{1}{3}\right)$$

$$= \frac{1}{27}(-2(2+4) + 2(-2-4) - (4-1))$$

$$= -1$$

and so the formula is checked.

4. Page 111, Exercise 1.

Let $F = T_aA$, $G = T_bB$. Then, by the result of Exercise III.1.2, FG has orthogonal part AB. Hence

$$\operatorname{sgn}(FG) = \det(AB), \quad \text{by definition,}$$

$$= (\det A)(\det B)$$

$$= (\operatorname{sgn} F)(\operatorname{sgn} G), \text{ by definition of } \operatorname{sgn} F, \operatorname{sgn} G.$$

Similarly

$$\operatorname{sgn}(GF) = (\operatorname{sgn} G)(\operatorname{sgn} F)$$

$$= (\operatorname{sgn} F)(\operatorname{sgn} G).$$

Putting $G = F^{-1}$, gives

$$\operatorname{sgn}(FF^{-1}) = (\operatorname{sgn} F)(\operatorname{sgn} F^{-1}).$$

But $FF^{-1} = I$, the identity mapping, and so

$$(\operatorname{sgn} F)(\operatorname{sgn} F^{-1}) = \operatorname{sgn} I = +1:$$

therefore

$$\operatorname{sgn} F^{-1} = \frac{1}{\operatorname{sgn} F}$$

$$= \operatorname{sgn} F, \qquad \text{because } \operatorname{sgn} F = \pm 1.$$

5. Page 111, Exercise 2.

Let G be an orientation-reversing isometry of E^3. We want to express G as H_0F, so we define F to be $H_0^{-1}G$. Then certainly

$$H_0F = H_0 H_0^{-1}G = G$$

and so all that remains to be shown is that F is orientation-preserving. Now, by the result of the previous exercise,

$$\operatorname{sgn} G = \operatorname{sgn}(H_0F) = (\operatorname{sgn} H_0)(\operatorname{sgn} F).$$

Here G and H_0 are both orientation-reversing, and so

$$\operatorname{sgn} G = \operatorname{sgn} H_0 = -1;$$

hence $\operatorname{sgn} F = +1$ and so F is orientation-preserving.

III.4 EUCLIDEAN GEOMETRY

Introduction

This section depends on Sections III.1 –III.3 and Section II.3. Now we are able to use the ideas and results of Sections III.1 – III.3 to begin to define what we mean by "the same, but for position in space, as . . .". Loosely speaking, we are going to declare two "objects" in E^3 (for example, curves or vector fields) to be "the same but for position in space" if there is an isometry carrying one into the other. In order that this definition agree with our intuitive ideas of "sameness", two conditions must hold: (a) the relation "is the same, but for position in space as" must be an equivalence relation (any sensible definition of "same" must lead to an equivalence relation); (b) any properties of the objects concerned that we regard as essential, such as the curvature of a curve, the derivative of a vector field, must be preserved by isometries, so that objects which are "the same" have the same essential properties. Condition (a) is easily checked: the relation is transitive because the composite of isometries is itself an isometry (Lemma III.1.3); it is reflexive because I is an isometry; it is symmetric because each isometry has an inverse isometry (Comment (i) on Section III.1). Condition (b) is the concern of this section, where it is shown that isometries preserve derivatives of vector fields on curves and the Frenet apparatus of unit-speed curves.

READ: Section III.4 (pages 112–115).

Additional Text

(i) *Arbitrary-Speed Curves and Isometries* While it is true that any mapping F preserves the velocity of a curve α in the sense that

$$F_*(\alpha') = (F(\alpha))'$$

(Theorem I.7.8), the speed of a curve is not generally preserved, because $\|\alpha'\|$ and $\|(F(\alpha))'\|$ are not necessarily the same. However, in the case where F is an isometry an argument similar to that given at the beginning of the proof of Theorem 4.2 shows that F does preserve speed: F preserves norms, as noted on page 105, and so

$$\begin{aligned} v_{F(\alpha)} &= \|(F(\alpha))'\| \\ &= \|F_*(\alpha')\| \\ &= \|\alpha'\| \\ &= v_\alpha . \end{aligned}$$

We have shown that α and $F(\alpha)$ have the same speed if F is an isometry. The following exercise shows that they have the same curvature, and plus-or-minus the same torsion.

1. Let α be an arbitrary-speed regular curve in E^3, F an isometry of E^3, $\beta = F(\alpha)$, and $\bar{\alpha}$ a unit-speed reparametrization of α, where $\alpha = \bar{\alpha}(s)$. Prove that $\kappa_\beta = \kappa_\alpha$ and $\tau_\beta = \operatorname{sgn} F\ \tau_\alpha$. (HINT: Use the above result to prove that $\bar{\beta} = F(\bar{\alpha})$ is a unit-speed reparametrization of β in the sense of Theorem II.2.1, and then use Theorem III.4.2.)

Supplementary Comments

(i) *Page 113: second half* As usual we are writing $F = TC$ and $C = (c_{ij})$.

(ii) *Page 114: fourth line of proof* T and $\bar{T}$ are defined by $T = \beta'$, $\bar{T} = \bar{\beta}'$, so the result follows using the fact that $\bar{\beta}' = (F(\beta))' = F_*(\beta')$ by Theorem I.7.8.

(iii) *Page 114: seventh line of proof* κ and $\bar{\kappa}$ are defined by $\kappa = \|\beta''\|$, $\bar{\kappa} = \|\bar{\beta}''\|$; $\bar{\beta}'' = F_*(\beta'')$ by Corollary III.4.1; $\|F_*(\beta'')\| = \|\beta''\|$ because F_* preserves norms.

(iv) *Page 114: eleventh line of proof* $(F_*(\beta''))/\kappa = F_*(\beta''/\kappa)$ because F_* is a linear transformation at each point.

Summary

Results

(i)	Isometries preserve derivatives of vector fields on curves: $(F_*(Y))' = F_*(Y')$. In particular, isometries preserve acceleration.	Page 113, Corollary 4.1
(ii)	Isometries preserve the speed of curves.	Text, Page 26
(iii)	Isometries preserve the Frenet apparatus of unit-speed curves, except that orientation-reversing isometries change the sign of the torsion and reverse the direction of the binormal: $\bar{\kappa} = \kappa$ $\quad \bar{T} = F_*(T)$ $\bar{\tau} = \operatorname{sgn} F\, \tau$ $\quad \bar{N} = F_*(N)$ $\bar{B} = \operatorname{sgn} F\, F_*(B)$	Page 114, Theorem 4.2
(iv)	A similar result for arbitrary-speed curves: if $\beta = F(\alpha)$, then $\kappa_\beta = \kappa_\alpha$ and $\tau_\beta = \operatorname{sgn} F\, \tau_\alpha$.	Text, Page 26

Techniques

(i)	Applying Result (i) to find the derivatives of vector fields of the form $F_*(Y)$ on a curve $F(\alpha)$.	Page 113, Corollary 4.1
(ii)	Using Result (iii) to find the Frenet apparatus of curves of the form $F(\alpha)$.	Page 114, Theorem 4.2

Exercises

Technique (i)

2. Page 115, Exercise 2. (Remember that, because C is orthogonal, C_* has the same matrix as C.)

Technique (ii)

3. Page 115, Exercise 1.

Solutions

1. We have seen that α and β have the same speed, and consequently α and β have the same arc-length function s. Now

$$\bar{\beta}(s) = (F(\bar{\alpha}))(s) = F(\bar{\alpha}(s)) = F(\alpha) = \beta$$

and so $\bar{\beta}$ is a unit-speed reparametrization of β in the sense of Theorem II.2.1.

Hence

$$\kappa_\beta = \kappa_{\bar{\beta}}(s) \qquad \tau_\beta = \tau_{\bar{\beta}}(s)$$

by the definition of the curvature and torsion of an arbitrary-speed curve. Similarly

$$\kappa_\alpha = \kappa_{\bar{\alpha}}(s) \qquad \tau_\alpha = \tau_{\bar{\alpha}}(s).$$

However, $\bar{\alpha}$ is unit-speed and $\bar{\beta} = F(\bar{\alpha})$ and so Theorem III.4.2. gives

$$\kappa_{\bar{\beta}} = \kappa_{\bar{\alpha}} \qquad \tau_{\bar{\beta}} = \operatorname{sgn} F\, \tau_{\bar{\alpha}}.$$

Hence

$$\kappa_\beta = \kappa_{\bar{\beta}}(s) = \kappa_{\bar{\alpha}}(s) = \kappa_\alpha$$

and

$$\tau_\beta = \tau_{\bar{\beta}}(s) = \operatorname{sgn} F\, \tau_{\bar{\alpha}}(s) = \operatorname{sgn} F\, \tau_\alpha.$$

2. Page 115, Exercise 2.

$$\bar{\alpha}(t) = C(\alpha(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} -\cos t \\ \frac{1}{\sqrt{2}}(\sin t - 2t) \\ \frac{1}{\sqrt{2}}(\sin t + 2t) \end{pmatrix}$$

$\overline{Y}(t) = C_*(Y(t))$: since C is itself an orthogonal transformation C_* has the same matrix as C and so

$$C_*(Y(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t \\ 1-t^2 \\ 1+t^2 \end{pmatrix}$$

$$= \begin{pmatrix} -t \\ -\sqrt{2}t^2 \\ \sqrt{2} \end{pmatrix}$$

That is,

$$\overline{Y}(t) = (-t, -\sqrt{2}t^2, \sqrt{2}).$$

Thus

$$\overline{Y}'(t) = (-1, -2\sqrt{2}t, 0).$$

Now

$$Y'(t) = (1, -2t, 2t)$$

and so

$$C_*(Y'(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -2\sqrt{2}t \\ 0 \end{pmatrix}$$

Thus $\overline{Y}'(t) = C_*(Y'(t))$ for all t, and so $\overline{Y}' = C_*(Y')$.

$$\alpha'(t) = (-\sin t, \cos t, 2),$$

$$\alpha''(t) = (-\cos t, -\sin t, 0)$$

so

$$C_*(\alpha''(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\frac{1}{\sqrt{2}} \sin t \\ -\frac{1}{\sqrt{2}} \sin t \end{pmatrix}$$

whereas

$$\bar{\alpha}'(t) = \left(\sin t, \frac{1}{\sqrt{2}}(\cos t - 2), \frac{1}{\sqrt{2}}(\cos t + 2)\right)$$

$$\bar{\alpha}''(t) = \left(\cos t, -\frac{\sin t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}\right)$$

and so $C_*(\alpha''(t)) = \bar{\alpha}''(t)$ for all t; that is, $C_*(\alpha'') = \bar{\alpha}''$.

We have already computed Y', α'', $\bar{Y}'$, $\bar{\alpha}''$: from this we see that

$$\begin{aligned}(Y'\cdot\alpha'')(t) &= (1, -2t, 2t)\cdot(-\cos t, -\sin t, 0) \\ &= -\cos t + 2t \sin t\end{aligned}$$

and

$$\begin{aligned}(\bar{Y}'\cdot\bar{\alpha}'')(t) &= (-1, -2\sqrt{2}t, 0)\cdot\left(\cos t, \frac{-\sin t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}\right) \\ &= -\cos t + 2t \sin t\end{aligned}$$

so

$$Y'\cdot\alpha'' = \bar{Y}'\cdot\bar{\alpha}''.$$

3. Page 115, Exercise 1.

(a) Since β has unit speed, $T = \beta'$, and so if β is a cylindrical helix then there is a constant unit vector **u** and a constant angle ϑ such $\beta'\cdot\mathbf{u} = \cos\vartheta$. Let $\bar{\mathbf{u}} = F_*(\mathbf{u})$. Then

$$\|\bar{\mathbf{u}}\| = \|F_*(\mathbf{u})\| = \|\mathbf{u}\| = 1$$

and so $\bar{\mathbf{u}}$ is a constant unit vector. Moreover, $F(\beta)$ also has unit speed; that is, $\bar{T} = F(\beta)'$, and so

$$\begin{aligned}\bar{T}\cdot\bar{\mathbf{u}} &= (F(\beta))'\cdot\bar{\mathbf{u}} \\ &= (F(\beta))'\cdot F_*(\mathbf{u}) \\ &= F_*(\beta')\cdot F_*(\mathbf{u}), && \text{by Theorem I.7.8,} \\ &= \beta'\cdot\mathbf{u}, && \text{because } F_* \text{preserves dot products,} \\ &= \cos\vartheta,\end{aligned}$$

and so $F(\beta)$ is also a cylindrical helix.

(b) Let $\overset{\frown}{F(\beta)}$ denote the spherical image of $F(\beta)$, $\overline{T}$ the unit tangent vector field on $F(\beta)$. Then $\overset{\frown}{F(\beta)}$ is the curve whose points have coordinates given by $\overline{T}$, while $\widetilde{\beta}$ is the curve whose points have coordinates given by T. In other words

$$T(t) = \widetilde{\beta}(t)_{\beta(t)} \quad \text{and} \quad \overline{T}(t) = \overset{\frown}{F(\beta)}(t)_{F(\beta(t))}.$$

Omitting the t:

$$T = \widetilde{\beta}_{\beta} \quad \text{and} \quad \overline{T} = \overset{\frown}{F(\beta)}_{F(\beta)}.$$

By Theorem III.4.2, $\overline{T} = F_*(T)$

and so $\overset{\frown}{F(\beta)}_{F(\beta)} = F_*(\widetilde{\beta}_{\beta})$.

Since $F = TC$, Theorem III.2.1 tells us that

$$F_*(\widetilde{\beta}_{\beta}) = C(\widetilde{\beta})_{F(\beta)}$$

and so

$$\overset{\frown}{F(\beta)}_{F(\beta)} = C(\widetilde{\beta})_{F(\beta)}$$

and so

$$\overset{\frown}{F(\beta)} = C(\widetilde{\beta}).$$

III.5 CONGRUENCE OF CURVES

Introduction

At last we reach the definition of congruence of curves promised in Section III.1 This is the definition suggested in Section III.4 and, as the remarks there show, congruence is in fact an equivalence relation. Results from all the earlier sections of Chapter III are used to prove the main theorem that unit-speed curves are congruent if and only if their curvatures are the same and their torsions differ at most in sign. Using the definitions of Section II.4 we can widen this theorem to include arbitrary-speed regular curves. Finally we generalize the main theorem to non-Frenet frame fields, using the ideas of Section II.6. This last theorem (Theorem III.5.7) is less important now but will be needed in Chapter VI: you are not expected to remember its details.

Some of the proofs in this section are a little long and technical. Try to work through them, but if you get bogged down make sure that you at least understand the statements of the theorems: they are the whole purpose of this chapter.

READ: Section III.5 (pages 116–121).

Comments

(i) *Pages 117–118: proof of Theorem 5.3* Notice the structure of the proof. If F is to carry α to β then certainly $F(\alpha(t)) = \beta(t)$ for all t, so our first requirement is that $F(\alpha(0)) = \beta(0)$. Secondly, we know from Theorem III.4.2 that F_* must take the Frenet frame of α at $\alpha(0)$ to the Frenet of β at $\beta(0)$, reversing the direction of the binormal if $\tau_\alpha = -\tau_\beta$, so we require that F_* have this effect on $T_\alpha(0)$, $N_\alpha(0)$, $B_\alpha(0)$. These two conditions specify F uniquely, so the final stage is to check that $F(\alpha)$ is actually equal to β.

Notice also the beauty and economy of the last stage of the proof. We know that $\bar{T}\cdot T \leqslant 1$ because

$$\bar{T}\cdot T = \|\bar{T}\|\,\|T\| \cos\vartheta,$$

where ϑ is the angle between $\bar{T}$ and T, and $\cos\vartheta = 1$ if and only if ϑ is a multiple of 2π, that is if and only if $\bar{T}$ and T are equal. Thus we can show that $\bar{T} = T$ by showing that $\bar{T}\cdot T$ is always 1. However, because the same remarks apply to $\bar{N}$ and N, $\bar{B}$ and B, we can show that all *three* pairs are equal by the *single* step of showing that

$$f = \bar{T}\cdot T + \bar{N}\cdot N + \bar{B}\cdot B = 3.$$

This is shown by a technique we have noted before, that of proving $f' = 0$ to show f is constant and noting the f(0) has the correct value.

(ii) *Page 119: Corollary 5.5* When O'Neill uses the word "helix" unqualified, he means not a general cylindrical helix, but a *circular* helix, that is a cylindrical helix whose cross-section curve is (part of) a circle (see Exercises II.4.8 and II.4.10), as he remarks in Example I.4.2(2). An equivalent definition of circular helix is as a curve congruent to a reparametrization of the helix

$$t \longmapsto (a\cos t,\ a\sin t,\ bt)$$

given in Example I.4.2(2), and it is this definition O'Neill uses in the proof here.

Supplementary Comments

(i) *Page 117: line −9* F is orientation-preserving because it carries one Frenet frame into another, and Frenet frames are always positively oriented.

(ii) *Page 117: line −2*

$$\begin{aligned} \bar{\kappa} &= \kappa_\alpha, && \text{by Theorem III.4.2,} \\ &= \kappa_\beta, && \text{by the hypotheses of the theorem.} \end{aligned}$$

(iii) *Page 119: Example 5.4* The "a" used here is simply a shorthand for $1/\sqrt{2} = 1/c$: it is not the same "a" as in Example II.3.3.

(iv) *Page 120: first two lines of proof* $\bar{\alpha}$ and $\bar{\beta}$ must be based at the same value of t so that α and β have the same arc-length function s, for if $\bar{\alpha}$ is based at t_1 and $\bar{\beta}$ at t_2 then

$$\begin{aligned} s_\alpha(t) &= \int_{t_1}^{t} v_\alpha(u)du \\ &= \int_{t_1}^{t} v_\beta(u)du, && \text{because } v_\alpha = v_\beta, \\ &= \int_{t_1}^{t_2} v_\beta(u)du + \int_{t_2}^{t} v_\beta(u)du \\ &= s_0 + s_\beta(t), \end{aligned}$$

and the constant s_0 is zero if and only if $t_1 = t_2$.

(v) *Page 121: Theorem 5.7, hypotheses (1) and (2)* These are just natural extensions of the hypotheses in Corollary 5.6, for if $\{E_i\}$ and $\{F_i\}$ are the Frenet frame fields on α, β respectively then condition (1) becomes simply $v_\alpha = v_\beta$ and condition (2) becomes $\kappa_\alpha = \kappa_\beta$, $\tau_\alpha = \tau_\beta$.

(vi) *Page 121: lines 7; 8 of proof* Since F carries the E_i frame at $\alpha(0)$ to the F_i frame at $\beta(0)$, the two left-hand equations are straightforward. For the other two:

$$\begin{aligned} \bar{\alpha}' \cdot \bar{E}_i &= (F(\alpha))' \cdot F_*(E_i), && \text{by definition,} \\ &= F_*(\alpha') \cdot F_*(E_i), && \text{because } F_* \text{ preserves velocities,} \\ &= \alpha' \cdot E_i, && \text{because } F_* \text{ preserves dot products,} \\ &= \beta' \cdot F_i, && \text{by hypothesis (1);} \end{aligned}$$

also

$$\begin{aligned} \bar{E}_i' \cdot \bar{E}_j &= (F_*(E_i))' \cdot F_*(E_j), && \text{by definition,} \\ &= F_*(E_i') \cdot F_*(E_j), && \text{because } F_* \text{ preserves derivatives of vector fields,} \\ &= E_i' \cdot E_j, && \text{because } F_* \text{ preserves dot products,} \\ &= F_i' \cdot F_j, && \text{by hypothesis (2).} \end{aligned}$$

(vii) *Page 121: line 9 of proof* Since $\{\overline{E}_i\}$ is a frame, $\overline{E}_i'$ has an orthonormal expansion $\overline{E}_i = \sum_j a_{ij}\overline{E}_j$, where $a_{ij} = \overline{E}_i'\cdot\overline{E}_j$ (Theorem II.1.5). Similarly, $F_i' = \sum_j b_{ij}F_j$, where $b_{ij} = F_i'\cdot F_j$. We have just shown that $F_i'\cdot F_j = \overline{E}_i'\cdot\overline{E}_j$, so $b_{ij} = a_{ij}$ and we can write $F_i' = \sum_j a_{ij}F_j$.

(viii) *Page 121: lines 10, 11 of proof* $\overline{E}_i\cdot\overline{E}_j = \delta_{ij} \Longrightarrow (\overline{E}_i\cdot\overline{E}_j)' = 0$

$$\Longrightarrow \overline{E}_i'\cdot\overline{E}_j + \overline{E}_i\cdot\overline{E}_j' = 0 \Longrightarrow a_{ij} + a_{ji} = 0.$$

(ix) *Page 121: line 13 of proof*

$$\begin{aligned} f' &= \sum_i (\overline{E}_i'\cdot F_i + \overline{E}_i\cdot F_i') \\ &= \sum_i \overline{E}_i'\cdot F_i + \sum_j \overline{E}_j\cdot F_j' \\ &= \sum_{i,j} (a_{ij}\overline{E}_j\cdot F_i + a_{ji}\overline{E}_j\cdot F_i) \\ &= \sum_{i,j} (a_{ij} + a_{ji})\, \overline{E}_j\cdot F_i. \end{aligned}$$

(x) *Page 121: line 14 of proof* For $i = 1, 2, 3$, since $\overline{E}_i(t)$ and $F_i(t)$ are unit vectors, $\overline{E}_i(t)\cdot F_i(t) \leqslant 1$ with equality if and only if $\overline{E}_i(t)$ and $F_i(t)$ are equal. Thus $f(t) = \sum_i \overline{E}_i(t)\cdot F_i(t) \leqslant 3$ with equality if and only if each $\overline{E}_i(t)$ is equal to the corresponding $F_i(t)$. We have shown that f' is the zero function, so f itself is a constant function. However, $f(0) = 3$ (because $\overline{E}_i(0) = F_i(0)$ for each i) and so $f(t) = 3$ for all t, that is $\overline{E}_i(t)$ is equal to $F_i(t)$ for all t.

Summary

Definitions

(i)	Congruent curves	Page 116, Definition 5.1
(ii)	Parallel curves	Page 116, line −1

Results

(i)	If $\alpha'(s)$ and $\beta'(s)$ are parallel for each s, α and β are parallel.	Page 117, Lemma 5.2
(ii)	If α and β are parallel and $\alpha(s_0) = \beta(s_0)$ for some s_0, then $\alpha = \beta$.	Page 117, Lemma 5.2
(iii)	Unit-speed curves α, β are congruent if and only if $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$.	Page 117, Theorem 5.3 and Page 114, Theorem 4.2
(iv)	Arbitrary-speed regular curves α, β are congruent if and only if $v_\alpha = v_\beta$, $\kappa_\alpha = \kappa_\beta$, $\tau_\alpha = \pm\tau_\beta$.	Page 120, Corollary 5.6 and Text, page 26

(v) If α, β are arbitrary curves and $\{E_i\}$, $\{F_i\}$ are frame fields on each such that

$$\alpha' \cdot E_i = \beta' \cdot F_i$$

and $$E_i' \cdot E_j = F_i' \cdot F_j,$$

then α and β are congruent. Page 121, Theorem 5.7

Techniques

(i) Recognition of congruent curves. Page 116, Definition 5.1

(ii) Finding an isometry establishing the congruence of two curves. Page 117, proof of Theorem 5.3

Exercises

Techniques (i) and (ii)

1. Page 122, Exercise 6.

Theory Exercise

2. Page 122, Exercise 5.

Solutions

1. Page 122, Exercise 6.

$$\alpha(t) = (\sqrt{2}t, t^2, 0)$$
$$\alpha'(t) = (\sqrt{2}, 2t, 0)$$
$$v_\alpha(t) = \sqrt{2}(1 + 2t^2)^{\frac{1}{2}}$$
$$\alpha''(t) = (0, 2, 0)$$

We use the formulas of Theorem II.4.3.

$$\alpha'(t) \times \alpha''(t) = (0, 0, 2\sqrt{2})$$

so

$$\kappa_\alpha(t) = \frac{2\sqrt{2}}{2\sqrt{2}(1 + 2t^2)^{\frac{3}{2}}} = \frac{1}{(1 + 2t^2)^{\frac{3}{2}}}.$$

$$\alpha'''(t) = 0 \quad \text{so} \quad \tau_\alpha(t) = 0.$$

$$\beta(t) = (-t, t, t^2)$$
$$\beta'(t) = (-1, 1, 2t)$$
$$v_\beta(t) = (2 + 4t^2)^{\frac{1}{2}} = \sqrt{2}(1 + 2t^2)^{\frac{1}{2}} = v_\alpha(t)$$
$$\beta''(t) = (0, 0, 2)$$
$$\beta'(t) \times \beta''(t) = (2, 2, 0)$$

so

$$\kappa_\beta(t) = \frac{2\sqrt{2}}{2\sqrt{2}(1 + 2t^2)^{\frac{3}{2}}} = \frac{1}{(1 + 2t^2)^{\frac{3}{2}}} = \kappa_\alpha(t).$$

$$\beta'''(t) = 0 \quad \text{so} \quad \tau_\beta(t) = 0 = \tau_\alpha(t).$$

Thus α and β have the same speed, curvature and torsion and so they are congruent.

$$\alpha'(0) = (\sqrt{2}, 0, 0) \quad \text{so} \quad T_\alpha(0) = (1, 0, 0);$$

$$\alpha'(0) \times \alpha''(0) = (0, 0, 2\sqrt{2}) \quad \text{so} \quad B_\alpha(0) = (0, 0, 1);$$

$$N_\alpha(0) = B_\alpha(0) \times T_\alpha(0) = (0, 1, 0):$$

$$\beta'(0) = (-1, 1, 0) \quad \text{so} \quad T_\beta(0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$\beta'(0) \times \beta''(0) = (2, 2, 0) \quad \text{so} \quad B_\beta(0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$N_\beta(0) = B_\beta(0) \times T_\beta(0) = (0, 0, 1).$$

Since $\tau_\alpha = \tau_\beta$ we want to find the isometry F carrying the α Frenet frame to the β Frenet frame: such an isometry has orthogonal part with matrix

$$C = \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}}_{\text{transpose of attitude matrix of } \beta \text{ frame}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{attitude matrix of } \alpha \text{ frame}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

Since $\alpha(0) = \beta(0) = \mathbf{0}$, $C(\alpha(0)) = \beta(0)$ and so the translation part of the F is just the identity.

Checking:

$$F(\alpha(t)) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}t \\ t^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t^2 \end{pmatrix} = \beta(t).$$

Alternatively, since $\tau_\alpha = -\tau_\beta$ also, we could choose F to take the α Frenet frame to the frame $T_\beta(0)$, $N_\beta(0)$, $-B_\beta(0)$. This gives

$$C = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

and again the translation part of F is the identity.

2. Page 122, Exercise 5.

If α and β are congruent, then $\kappa_\alpha = \kappa_\beta = \kappa$ and $\tau_\alpha = \pm\,\tau_\beta = \tau$. Suppose F is an isometry such that $F(\alpha) = \beta$. Then, as noted in the Additional Text (i), we must have

$$\begin{aligned} F(\alpha(0)) &= \beta(0) \\ F_*(T_\alpha(0)) &= T_\beta(0) \\ F_*(N_\alpha(0)) &= N_\beta(0) \\ F_*(B_\alpha(0)) &= \operatorname{sgn} F\, B_\beta(0) \qquad (*) \end{aligned}$$

by Theorem III.4.2. This theorem also tells us that

$$\tau_\beta = \operatorname{sgn} F\, \tau_\alpha.$$

If $\tau \neq 0$, this equation tells us that $\operatorname{sgn} F = 1$ if $\tau_\alpha = \tau_\beta$, $\operatorname{sgn} F = -1$ if $\tau_\alpha = -\tau_\beta$. Thus the four equations (*) determine the isometry F uniquely, and the proof of Theorem III.5.3 shows that in fact $F(\alpha) = \beta$.

If $\tau = 0$ then there is no constraint on sgn F, so we can have $\operatorname{sgn} F = 1$ or $\operatorname{sgn} F = -1$. In each of these two cases the equations (*) give a unique isometry F. Moreover, since $\tau_\alpha = \tau_\beta$ and $\tau_\alpha = -\tau_\beta$, the proof of Theorem III.5.3 shows that both of these isometries satisfy $F(\alpha) = \beta$.

FURTHER EXERCISES AND SOLUTIONS

Section III.1

Technique Exercises

Page 103, Exercises 5(b), (c).

Solutions

5(b). $(5\sqrt{2}, -5, 4\sqrt{2})$

5(c). $(-2\sqrt{2}, 1, 5\sqrt{2})$.

Section III.2

Technique Exercise

Page 107, Exercise 4(b). (HINT: $(0, 1, 0)$ has the same length as $(1, 0, -1)/\sqrt{2}$.)

Other Recommended Exercise

Page 106, Exercise 3. This proves the result given at the end of page 106 in an alternative way.

Solutions

4(b). There are many possible solutions, each of which is obtained by putting a particular value of ϑ in the following.

$$C = \begin{pmatrix} -\dfrac{\sin\vartheta}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & \dfrac{\cos\vartheta}{\sqrt{2}} \\ \cos\vartheta & 0 & \sin\vartheta \\ -\dfrac{\sin\vartheta}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & \dfrac{\cos\vartheta}{\sqrt{2}} \end{pmatrix}$$

and T is translation through

$$\left(1 + \frac{\sin\vartheta}{2\sqrt{2}} + \frac{1}{\sqrt{2}}, -2 - \frac{\cos\vartheta}{2}, 1 + \frac{\sin\vartheta}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right).$$

3(a). The translation part of F carries $C(\mathbf{0}) = \mathbf{0}$ to $\mathbf{p}$, so is $T_{\mathbf{p}}$.

The orthogonal part of F carries $U_i(\mathbf{0})$ to $\mathbf{e}_i$, so has the coordinates of the $\mathbf{e}_i$ as its *columns*, so it is ${}^tA$.

3(b). The e-frame at p is taken to the natural frame at 0 by $(T_pA^{-1})^{-1} = AT_p^{-1} = T_{A(-p)}A$.

The natural frame at 0 is taken to the f-frame at q by T_qB^{-1}.

$(T_qB^{-1})(T_{A(-p)}A)$ takes the e-frame at p to the f-frame at q and has orthogonal part $B^{-1}A$.

Section III.3

Recommended Exercises

Page 111, Exercise 4. This shows that orthogonal transformations with determinant +1 can be realized as rotations of E^3.

Page 111, Exercise 6. Some more group theory.

Page 111, Exercise 7. This looks at isometries of E^1 and E^2, which are defined in the obvious way.

Solutions

4. Put $e_3 = (p/\|p\|)$. Choose e_1 and e_2 to be orthogonal unit vectors spanning the subspace of E^3 orthogonal to e_3. $C(e_1)$ is also orthogonal to e_3 and has unit length, so

$$C(e_1) = \cos\vartheta\ e_1 + \sin\vartheta\ e_2$$

for some ϑ. If

$$C(e_2) = \sin\vartheta\ e_1 - \cos\vartheta\ e_2,$$

replace ϑ and e_2 by their negatives.

6. For both parts: $\det(AB) = \det A \det B$

$$\det A^{-1} = (\det A)^{-1}$$

$$\det I = +1.$$

7. E^1: $F(x) = \epsilon x + a$

$$E^2\text{: } F = TC,\ C = \begin{pmatrix} \cos\vartheta & \epsilon\sin\vartheta \\ -\sin\vartheta & \epsilon\cos\vartheta \end{pmatrix}.$$

In each case, orientation-preserving if $\epsilon = +1$.

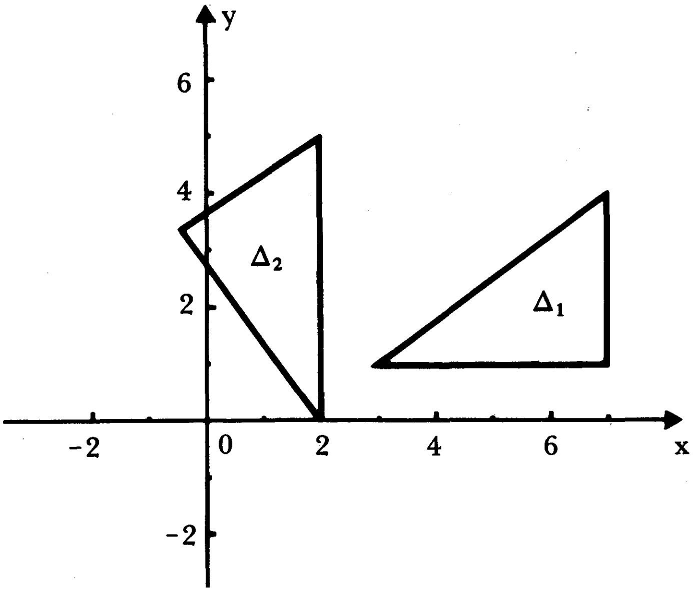
Section III.4

Technique Exercise

Page 116, Exercise 3.

Solution

3.



$$T_{(3,-3)}\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Section III.5

Technique Exercises

Page 122, Exercises 2,4. (HINT for Exercise 2: The Frenet apparatus is not needed – look for an isometry first.)

Other Recommended Exercises

Page 121, Exercise 1. This gives another characterization of congruence of curves.

Page 122, Exercise 3. This uses Theorem III.5.7 and the connection forms.

Page 122, Exercise 11. This proves a theorem equivalent to Theorem III.4.2 and Theorem III.5.3 for curves in the plane.

Solutions

2. There are eight possible solutions. Each of ϵ, δ, η may be ±1 in the following.

$$\gamma(t) = (\epsilon 3\sqrt{2}t, \delta 3t^2, \eta\sqrt{2}t^3)$$

$$F = \begin{pmatrix} \frac{\epsilon}{\sqrt{2}} & 0 & \frac{\epsilon}{\sqrt{2}} \\ 0 & \delta & 0 \\ -\frac{\eta}{\sqrt{2}} & 0 & \frac{\eta}{\sqrt{2}} \end{pmatrix}$$

4. $\kappa = \frac{1}{4}$, $\tau = -\frac{1}{4}$.

$a = 2$, $b = 2$.

$$F = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

1. If $F = T_pC$, where $C(u_i) = e_i$, then $F(\alpha) = \beta$.

3. The frame field on α is $\{E_i(\alpha)\}$, that on β is $\{E_i(\beta)\}$.

$$\alpha' \cdot E_i(\alpha) = \theta_i(\alpha').$$

$$(E_i(\alpha))' \cdot E_j(\alpha) = (\nabla_{\alpha'} E_i) \cdot E_j(\alpha) = \omega_{ij}(\alpha').$$

11. If: let F be the isometry taking

$\alpha(0)$ to $\beta(0)$

$T_\alpha(0)$ to $T_\beta(0)$

$N_\alpha(0)$ to $\epsilon N_\beta(0)$ where $\kappa_\beta = \epsilon\kappa_\alpha$.

Follow the proof of Theorem III.5.3.

Only if: follow the proof of Theorem III.4.2.

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3